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Oscillation and Nonoscillation for Second Order Linear Impulsive Differential Equations*

Chunchao Huang

Department of Mathematics, Fuzhou University, Fuzhou, Fujian, 350002,

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We study the oscillation and nonoscillation for the second order linear impulsive differential equation $u'' = -p(t)u$, where $p(t)$ is an impulsive function defined by $p(t) = \sum_{n=1}^{\infty} a_n \delta(t - t_n)$, and we establish a necessary and sufficient condition for oscillation (or nonoscillation) of the equation $u'' = -p(t)u$. We give some sufficient conditions for the oscillation and nonoscillation in two particular cases.

1. INTRODUCTION

Let $\delta(t)$ be a δ-function, i.e., $\int_{-\infty}^{\infty} \delta(t)\varphi(t)\,dt = \int_{-\varepsilon}^{\varepsilon} \delta(t)\varphi(t)\,dt = \varphi(0)$ for all $\varphi(t)$ being continuous at $t = 0$, and set

$$p(t) = \sum_{n=1}^{\infty} a_n \delta(t - t_n), \tag{1.1}$$

where $0 \le t_0 < t_1 < t_2 < \cdots < t_n < \cdots$, and $t_n \to \infty$ as $n \to \infty$, and $a_n > 0$ for all $n \in N$.

In this paper we discuss the oscillatory and nonoscillatory properties of the second order linear impulsive differential equation

$$u'' = -p(t)u, \qquad t \ge 0. \tag{1.2}$$

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The function $u(t)$ is called a solution of (1.2) if it is continuous on $[0, \infty)$, and it is a linear function on every interval $[t_n, t_{n+1}]$, and $u'(t_n^-) - u'(t_n^+) = \lim_{\varepsilon \to 0^+} \int_{t_n-\varepsilon}^{t_n+\varepsilon} p(t)u(t)\,dt = u(t_n)a_n$ for all $n \in N$, where $u'(t_n^+), u'(t_n^-)$ denote the right-derivative and the left-derivative of $u(t)$ at the point t_n, respectively. So, if $u(t)$ is a solution of (1.2), then

$$u'(t_n^+) = u'(t) = u'(t_{n+1}^-), \qquad \text{for } t_n < t < t_{n+1}, \tag{1.3}$$

$$u'(t_n^+) - u'(t_{n+1}^+) = u'(t_{n+1}^-) - u'(t_{n+1}^+) = u(t_{n+1})a_{n+1}, \tag{1.4}$$

and

$$\begin{aligned} u(t_{n+1}) - u(t) &= u'(t_n^+)(t_{n+1} - t) \\ &= u'(t_{n+1}^-)(t_{n+1} - t), \qquad \text{for } t_n \le t < t_{n+1} \end{aligned} \tag{1.5}$$

for all $n \in N$.

A nontrivial solution $u(t)$ of (1.2) is called oscillatory if it has arbitrarily large zeros and is nonoscillatory otherwise. Equation (1.2) is said to be nonoscillatory if all its nontrivial solutions are nonoscillatory, and to be oscillatory if all its nontrivial solutions are oscillatory.

For each $n \in N$ define the sequence $\{P_k^n\}_{k=1}^\infty$ by induction,

$$\begin{aligned} P_1^n &= a_{n+1}(t_{n+1} - t_n), \\ P_{k+1}^n &= \frac{a_{n+k+1}}{a_{n+k}} \left(\frac{P_k^n}{1 - P_k^n} + a_{n+k}(t_{n+k+1} - t_{n+k}) \right), \end{aligned}$$

where $P_{k+1}^n = \infty$ provided $P_k^n = 1$ and $P_{k+1}^n = 0$ provided $P_k^n = \infty$. Denote

$$S_n = \sup\{ P_k^n \mid k = 1, 2, \dots \}.$$

Remark 1. From the definition of P_k^n and S_n we see that $S_n \le 1$ implies $0 < P_k^n < 1$ for all $k \in N$.

The main results of the paper are as follows.

THEOREM 1. *The following statements are equivalent*:

(i) *There is $n_0 \in N$ such that $S_{n_0} \le 1$.*

(ii) *There is $n_0 \in N$ such that $S_n \le 1$ for all $n \ge n_0$.*

(iii) *Equation* (1.2) *is nonoscillatory.*

(iv) *Equation* (1.2) *has a nonoscillatory solution.*

Theorem 1 will be proved in Section 3. In Section 4 and Section 5 we apply Theorem 1 to discuss the nonoscillation and oscillation of (1.2) in the cases of $t_n = t_0 + \lambda^{n-1}T, \lambda > 1, T > 0$, and $t_n = t_0 + nT$, and establish the sufficient conditions for the nonoscillation and the oscillation of (1.2) in both cases separately.

2. TWO LEMMAS

LEMMA 1. *If $S_{n_0} \le 1$ for some $n_0 \in N$, then $S_{n_0+1} \le 1$, therefore $S_n \le 1$ for all $n \ge n_0$.*

Proof. In order to show this we prove (by induction)

$$0 < P_k^{n_0+1} \le P_{k+1}^{n_0} < 1, \qquad \text{for all } k \in N. \tag{2.1}$$

First, we notice that $S_{n_0} \le 1, a_k > 0$ and $t_{k+1} - t_k > 0$ for all $k \in N$ imply $0 < P_k^{n_0} < 1$ for all $k \in N$. Then

$$\begin{aligned} 0 < P_1^{n_0+1} &= a_{n_0+2}\left(t_{n_0+2} - t_{n_0+1}\right) \\ &< \frac{a_{n_0+2}}{a_{n_0+1}}\left(\frac{P_1^{n_0}}{1 - P_1^{n_0}} + a_{n_0+1}\left(t_{n_0+2} - t_{n_0+1}\right)\right) \\ &= P_2^{n_0}, \end{aligned}$$

which shows that (2.1) holds for $k = 1$.

We now assume by induction that (2.1) is true for $k = m$, i.e.,

$$0 < P_m^{n_0+1} \le P_{m+1}^{n_0} < 1. \tag{2.2}$$

In view of the fact that the function $f(x) = x/(1-x)$ is increasing (strictly) in $x \in [0, 1)$, then

$$\begin{aligned} 1 > P_{m+2}^{n_0} &= \frac{a_{n_0+m+2}}{a_{n_0+m+1}}\left(\frac{P_{m+1}^{n_0}}{1 - P_{m+1}^{n_0}} + a_{n_0+m+1}\left(t_{n_0+m+2} - t_{n_0+m+1)}\right)\right) \\ &\ge \frac{a_{n_0+m+2}}{a_{n_0+m+1}}\left(\frac{P_m^{n_0+1}}{1 - P_m^{n_0+1}} + a_{n_0+m+1}\left(t_{n_0+m+2} - t_{n_0+m+1}\right)\right) \\ &= P_{m+1}^{n_0+1} \\ &> 0, \end{aligned}$$

which shows that (2.1) also is true for $k = m + 1$. Hence (2.1) is true for all $k \in N$ by induction, therefore $S_{n_0+1} \le S_{n_0} \le 1$. Lemma 1 is proved. ∎

LEMMA 2. *Suppose that* $0 < \alpha_k \le \beta_k$, $\lambda_k > 0$ *for all* $k \in N$. *Define by induction*

$$p_1 = \alpha_1 \lambda_1, \qquad q_1 = \beta_1 \lambda_1,$$

$$p_{k+1} = \frac{\alpha_{k+1}}{\alpha_k}\left(\frac{p_k}{1-p_k} + \alpha_k \lambda_{k+1}\right),$$

$$q_{k+1} = \frac{\beta_{k+1}}{\beta_k}\left(\frac{q_k}{1-q_k} + \beta_k \lambda_{k+1}\right), \qquad k = 1, 2, \dots .$$

If $0 < q_k < 1$ *for all* $k \in N$, *then*

$$0 < p_k \le \frac{\alpha_k}{\beta_k} q_k < 1, \qquad \text{for all } k \in N. \tag{2.3}$$

Proof. For $k = 1$, it is clear that

$$0 < p_1 = \alpha_1 \lambda_1 = \frac{\alpha_1}{\beta_1} q_1.$$

We now assume by induction that (2.3) is true for $k = m$, i.e.,

$$0 < p_m \le \frac{\alpha_m}{\beta_m} q_m < 1. \tag{2.4}$$

In view of the increasing of $f(x) = x/(1-x)$ in $x \in [0, 1)$, then

$$\begin{aligned} 0 < p_{m+1} &= \frac{\alpha_{m+1}}{\alpha_m}\left(\frac{p_m}{1-p_m} + \alpha_m \lambda_{m+1}\right) \\ &\le \frac{\alpha_{m+1}}{\alpha_m}\left(\frac{(\alpha_m/\beta_m) q_m}{1-(\alpha_m/\beta_m) q_m} + \alpha_m \lambda_{m+1}\right) \\ &= \frac{\alpha_{m+1}}{\beta_m}\left(\frac{q_m}{1-(\alpha_m/\beta_m) q_m} + \beta_m \lambda_{m+1}\right) \\ &\le \frac{\alpha_{m+1}}{\beta_m}\left(\frac{q_m}{1-q_m} + \beta_m \lambda_{m+1}\right) \\ &= \frac{\alpha_{m+1}}{\beta_{m+1}} q_{m+1} < 1, \end{aligned}$$

which shows that (2.3) also is true for $k = m + 1$. Hence (2.3) is true for all $k \in N$ by induction, and Lemma 2 is proved. ∎

3. PROOF OF THEOREM 1

Proof of Theorem 1. It is clear from Lemma 1 that (i) implies (ii). Suppose that (ii) holds. Then there is $n_0 \in N$ such that

$$0 < P_k^n < 1, \qquad \text{for all } n \geq n_0,\, k \geq 1. \tag{3.1}$$

Let $u(t)$ be a nontrivial solution of (1.2). We assume without loss of generality that there exist $n \geq n_0$ and $\bar{t} \in [t_n, t_{n+1})$ such that

$$u(\bar{t}) = 0, \qquad u'(\bar{t}^+) > 0.$$

In order to show (iii) that $u(t)$ is nonoscillatory it suffices to show that

$$u'(t_{n+k}^+) > 0, \qquad \text{for all } k \in N.$$

From (1.3) and (1.4), we have

$$\begin{aligned} u'(\bar{t}^+) - u'(t_{n+1}^+) &= u'(t_{n+1}^-) - u'(t_{n+1}^+) \\ &= u(t_{n+1})a_{n+1} \\ &\leq u'(\bar{t}^+)(t_{n+1} - t_n)a_{n+1} \\ &= P_1^n u'(\bar{t}^+). \end{aligned}$$

Hence, by (3.1),

$$u'(t_{n+1}^+) \geq (1 - P_1^n)u'(\bar{t}^+) > 0, \tag{3.2}$$

and

$$\begin{aligned} \frac{P_1^n}{1 - P_1^n} u'(t_{n+1}^+) &\geq P_1^n u'(\bar{t}^+) \\ &\geq u(t_{n+1})a_{n+1}, \end{aligned}$$

hence

$$\begin{aligned} &\frac{a_{n+2}}{a_{n+1}}\left(\frac{P_1^n}{1 - P_1^n} + a_{n+1}(t_{n+2} - t_{n+1})\right)u'(t_{n+1}^+) \\ &\quad \geq \left(u(t_{n+1}) + u'(t_{n+1}^+)(t_{n+2} - t_{n+1})\right)a_{n+2} \\ &\quad = u(t_{n+2})a_{n+2}. \end{aligned}$$

Therefore, because of the definition of P_2^n,

$$P_2^n u'(t_{n+1}^+) \geq u(t_{n+2})a_{n+2}. \tag{3.3}$$

Now we prove by use of induction that the following two formulae are valid for all $k \in N$:

$$u'(t_{n+k}^{+}) \geq (1 - P_k^n) u'(t_{n+k-1}^{+}) > 0, \tag{3.4}$$

and

$$P_{k+1}^n u'(t_{n+k}^{+}) \geq u(t_{n+k+1}) a_{n+k+1}. \tag{3.5}$$

Since $t_n \leq \bar{t} < t_{n+1}$ implies $u'(t_n^{+}) = u'(\bar{t}^{+}) = u'(t_{n+1}^{-})$, then (3.2) and (3.3) show that (3.4) and (3.5) are true for $k = 1$. We now assume by induction that (3.4) and (3.5) are true for $k = m$, i.e.,

$$u'(t_{n+m}^{+}) \geq (1 - P_m^n) u'(t_{n+m-1}^{+}) > 0, \tag{3.6}$$

$$P_{m+1}^n u'(t_{n+m}^{+}) \geq u(t_{n+m+1}) a_{n+m+1}. \tag{3.7}$$

Then from (3.7) and (1.4) we get

$$\begin{aligned} P_{m+1}^n u'(t_{n+m}^{+}) &\geq u'(t_{n+m+1}^{-}) - u'(t_{n+m+1}^{+}) \\ &= u'(t_{n+m}^{+}) - u'(t_{n+m+1}^{+}). \end{aligned}$$

So

$$u'(t_{n+m+1}^{+}) \geq (1 - P_{m+1}^n) u'(t_{n+m}^{+}) > 0. \tag{3.8}$$

In view of (3.7) again, from (3.8) we obtain that

$$\frac{P_{m+1}^n}{1 - P_{m+1}^n} u'(t_{n+m+1}^{+}) \geq u(t_{n+m+1}) a_{n+m+1},$$

hence

$$\begin{aligned} &P_{m+2}^n u'(t_{n+m+1}^{+}) \\ &\quad = \frac{a_{n+m+2}}{a_{n+m+1}} \left(\frac{P_{m+1}^n}{1 - P_{m+1}^n} + a_{n+m+1}(t_{n+m+2} - t_{n+m+1}) \right) u'(t_{n+m+1}^{+}) \\ &\quad \geq u(t_{n+m+1}) a_{n+m+2} + a_{n+m+2}(t_{n+m+2} - t_{n+m+1}) u'(t_{n+m+1}^{+}) \\ &\quad = u(t_{n+m+2}) a_{n+m+2}. \end{aligned} \tag{3.9}$$

Inequalities (3.8) and (3.9) show that (3.4) and (3.5) also are true for $k = m + 1$, therefore (3.4) and (3.5) are valid for all $k \in N$ by induction. That is, (ii) implies (iii).

It is clear that (iii) implies (iv).

Finally, we prove that (iv) implies (i). Suppose that (iv) holds and let $u(t)$ be a nonoscillatory solution of (1.2). Without loss of generality we can assume that there is a t_{n_0} such that

$$u(t) > 0, \qquad \text{for } t > t_{n_0},$$

and so

$$u'\left(t_{n_0+k}^+\right) > 0, \qquad k = 0, 1, 2, \dots . \tag{3.10}$$

We now prove $S_{n_0} \leq 1$. In order to show this, it suffices to show that the following two relations are valid for all $k \in N$:

$$0 < P_k^{n_0}, u'\left(t_{n_0+k}^+\right) \leq \left(1 - P_k^{n_0}\right) u'\left(t_{n_0+k-1}^+\right), \tag{3.11}$$

and

$$P_{k+1}^{n_0} u'\left(t_{n_0+k}^+\right) \leq u\left(t_{n_0+k+1}\right) a_{n_0+k+1}. \tag{3.12}$$

When $k = 1$, it is obvious that

$$P_1^{n_0} > 0. \tag{3.13}$$

Now, since

$$\begin{aligned} u\left(t_{n_0+1}\right) &= u\left(t_{n_0}\right) + u'\left(t_{n_0}^+\right)\left(t_{n_0+1} - t_{n_0}\right) \\ &\geq u'\left(t_{n_0}^+\right)\left(t_{n_0+1} - t_{n_0}\right), \end{aligned}$$

hence, and because of (1.4),

$$\begin{aligned} u'\left(t_{n_0}^+\right) - u'\left(t_{n_0+1}^+\right) &= u\left(t_{n_0+1}\right) a_{n_0+1} \\ &\geq u'\left(t_{n_0}^+\right)\left(t_{n_0+1} - t_{n_0}\right) a_{n_0+1} \\ &= P_1^{n_0} u'\left(t_{n_0}^+\right). \end{aligned} \tag{3.14}$$

Therefore

$$u'\left(t_{n_0+1}^+\right) \leq \left(1 - P_1^{n_0}\right) u'\left(t_{n_0}^+\right). \tag{3.15}$$

Inequalities (3.10) and (3.15) imply $1 - P_1^{n_0} > 0$. Hence from (3.13), (3.14), and (3.15) we get

$$\begin{aligned} \frac{P_1^{n_0}}{1 - P_1^{n_0}} u'\left(t_{n_0+1}^+\right) &\leq P_1^{n_0} u'\left(t_{n_0}^+\right) \\ &\leq u\left(t_{n_0+1}\right) a_{n_0+1}, \end{aligned}$$

so

$$\begin{aligned}
&\frac{a_{n_0+2}}{a_{n_0+1}}\left(\frac{P_1^{n_0}}{1-P_1^{n_0}}+a_{n_0+1}\left(t_{n_0+2}-t_{n_0+1}\right)\right)u'\left(t_{n_0+1}^+\right)\\
&\qquad\leq\left(u\left(t_{n_0+1}\right)+u'\left(t_{n_0+1}^+\right)\left(t_{n_0+2}-t_{n_0+1}\right)\right)a_{n_0+2}\\
&\qquad=u\left(t_{n_0+2}\right)a_{n_0+2},
\end{aligned}$$

i.e.,

$$P_2^{n_0}u'\left(t_{n_0+1}^+\right)\leq u\left(t_{n_0+2}\right)a_{n_0+2}. \tag{3.16}$$

Inequalities (3.13), (3.15), and (3.16) show that (3.11) and (3.12) are true for $k=1$.

We now assume by induction that (3.11) and (3.12) are true for $k=m\geq 1$, i.e.,

$$0<P_m^{n_0},u'\left(t_{n_0+m}^+\right)\leq\left(1-P_m^{n_0}\right)u'\left(t_{n_0+m-1}^+\right), \tag{3.17}$$

and

$$P_{m+1}^{n_0}u'\left(t_{n_0+m}^+\right)\leq u\left(t_{n_0+m+1}\right)a_{n_0+m+1}; \tag{3.18}$$

(3.10) and (3.17) imply obviously that $0<P_m^{n_0}<1$, and hence

$$P_{m+1}^{n_0}>0. \tag{3.19}$$

In view of (1.4), the inequality (3.18) implies

$$P_{m+1}^{n_0}u'\left(t_{n_0+m}^+\right)\leq u'\left(t_{n_0+m}^+\right)-u'\left(t_{n_0+m+1}^+\right),$$

hence

$$u'\left(t_{n_0+m+1}^+\right)\leq\left(1-P_{m+1}^{n_0}\right)u'\left(t_{n_0+m}^+\right). \tag{3.20}$$

Therefore, and because of (3.18) and (3.19),

$$\frac{P_{m+1}^{n_0}}{1-P_{m+1}^{n_0}}u'\left(t_{n_0+m+1}^+\right)\leq u\left(t_{n_0+m+1}\right)a_{n_0+m+1},$$

consequently, because of the definition of P_{m+2}^{n}, and as before

$$\begin{aligned}
&P_{m+2}^{n_0}u'\left(t_{n_0+m+1}^+\right)\\
&\qquad\leq\left(u\left(t_{n_0+m+1}\right)+u'\left(t_{n_0+m+1}^+\right)\left(t_{n_0+m+2}-t_{m_0+m+1}\right)\right)a_{n_0+m+2}\\
&\qquad=u\left(t_{n_0+m+2}\right)a_{n_0+m+2}.
\end{aligned} \tag{3.21}$$

From (3.19), (3.20), and (3.21) we see that (3.11) and (3.12) also are true for $k = m + 1$, hence (3.11) and (3.12) are valid for all $k \in N$ by induction. From (3.10) and (3.11) we get that $0 < P_k^{n_0} < 1$ for all $k \in N$. Thus (i) holds. That is, (iv) implies (i).

The proof of Theorem 1 is accomplished. ∎

COROLLARY 1. *Let $q(t) = \sum_{n=1}^{\infty} b_n \delta(t - t_n)$. Suppose that $0 < a_n \le b_n$, for large $n \in N$. Then*

(i) *If equation*

$$u' = -q(t)u \tag{3.22}$$

is nonoscillatory, then (1.2) *is nonoscillatory.*

(ii) *If* (1.2) *is oscillatory, then* (3.22) *is oscillatory.*

Proof. (i) For each $n \in N$ define the sequence $\{q_k^n\}_{k=1}^{\infty}$ by induction: $q_1^n = b_{n+1}(t_{n+1} - t_n)$, $q_{k+1}^n = (b_{n+k+1}/b_{n+k})(q_k^n/(1 - q_k^n) + b_{n+k}(t_{n+R+1} - t_{n+k}))$. Since Eq. (3.22) is nonoscillatory, according to Theorem 1 there exists $n_0 \in N$ such that

$$0 < q_k^n < 1, \qquad \text{for all } k \in N, n \ge n_0. \tag{3.23}$$

According to the hypothesis there exists $n_1 > n_0$ such that

$$0 < a_{n_1+k} \le b_{n_1+k}, \qquad \text{for all } k \in N. \tag{3.24}$$

Thus from Lemma 2 we get that when $n \ge n_1$

$$0 < P_k^n \le \frac{a_{n+k}}{b_{n+k}} q_k^n \le q_k^n < 1, \qquad \text{for all } k \in N,$$

hence (1.2) is nonoscillatory by Theorem 1.

(ii) The proof is accomplished immediately from (i).

Corollary 1 is proved. ∎

COROLLARY 2. *Let $h(t)$ be nonnegative continuous on $[0, \infty)$, and $b_n = \int_{t_n}^{t_{n+1}} h(t)\, dt > 0$. Let*

$$g_1(t) = \sum_{n=1}^{\infty} b_n \delta(t - t_n),$$

$$g_2(t) = \sum_{n=1}^{\infty} b_n \delta(t - t_{n+1}).$$

Then

(i) *If* $u'' = -g_1(t)u$ *is oscillatory, then*

$$u'' = -h(t)u \tag{3.25}$$

is oscillatory.

(ii) *If* $u'' = -g_2(t)u$ *is nonoscillatory, then* (3.25) *is nonoscillatory.*

Proof. (i) The proof will be accomplished by contradiction. We suppose that Eq. (3.25) has a nonoscillatory solution $u(t)$. Without loss of generality we can assume that there is a t_{n_0} such that

$$u(t) > 0, \qquad u'(t) > 0, \text{ for all } t \geq t_{n_0}. \tag{3.26}$$

We define the sequence $\{q_k^n\}_{k=1}^{\infty}$ to be similar to the one in the proof of Corollary 1(i).

Fix $n > n_0$. Later we will prove that $0 < q_k^n < 1$ for all $k \in N$.

First, using (3.26) and the definition of b_n and q_k^n, we have

$$\begin{aligned} u'(t_{n+1}) - u'(t_{n+2}) &= \int_{t_{n+1}}^{t_{n+2}} h(t)u(t)\, dt \\ &\geq u(t_{n+1})b_{n+1} \\ &\geq \big(u(t_n) + u'(t_{n+1})(t_{n+1} - t_n)\big)b_{n+1} \\ &\geq q_1^n u'(t_{n+1}), \end{aligned}$$

hence

$$0 < u'(t_{n+2}) \leq (1 - q_1^n)u'(t_{n+1}).$$

Therefore

$$0 < q_1^n < 1, \tag{3.27}$$

$$\frac{q_1^n}{1 - q_1^n} u'(t_{n+2}) \leq q_1^n u'(t_{n+1}) \leq u(t_{n+1})b_{n+1}. \tag{3.28}$$

Now, we claim by induction that for all $k \in N$ the following two relations hold:

$$0 < q_k^n < 1, \tag{3.29}$$

$$\frac{q_k^n}{1 - q_k^n} u'(t_{n+k+1}) \leq u(t_{n+k})b_{n+k}. \tag{3.30}$$

In fact, (3.27) and (3.28) show that (3.29) and (3.30) are valid for $k = 1$. We now assume that (3.29) and (3.30) are true for $k = m$, i.e.,

$$0 < q_m^n < 1, \tag{3.31}$$

$$\frac{q_m^n}{1 - q_m^n} u'(t_{n+m+1}) \le u(t_{n+m}) b_{n+m}. \tag{3.32}$$

From (3.32) we get

$$\begin{aligned} \frac{b_{n+m+1}}{b_{n+m}} & \left(\frac{q_m^n}{1 - q_m^n} + b_{n+m}(t_{n+m+1} - t_{n+m}) \right) u'(t_{n+m+1}) \\ & \le \big(u(t_{n+m}) + u'(t_{n+m+1})(t_{n+m+1} - t_{n+m})\big) b_{n+m+1} \\ & \le u(t_{n+m+1}) b_{n+m+1}. \end{aligned} \tag{3.33}$$

According to the definition of q_{m+1}^n, (3.33) can be written as

$$q_{m+1}^n u'(t_{n+m+1}) \le u(t_{n+m+1}) b_{n+m+1}. \tag{3.34}$$

Hence, because of the definition of b_{n+m+1},

$$\begin{aligned} q_{m+1}^n u'(t_{n+m+2}) &\le u(t_{n+m+1}) \int_{t_{n+m+1}}^{t_{n+m+2}} h(t)\, dt \\ &\le \int_{t_{n+m+1}}^{t_{n+m+2}} u(t) h(t)\, dt \\ &= u'(t_{n+m+1}) - u'(t_{n+m+2}), \end{aligned}$$

therefore

$$0 < u'(t_{n+m+2}) \le (1 - q_{m+1}^n) u'(t_{n+m+1}). \tag{3.35}$$

In view of the fact that (3.31) implies $q_{m+1}^n > 0$, (3.26) and (3.35) come to

$$0 < q_{m+1}^n < 1, \tag{3.36}$$

$$\frac{q_{m+1}^n}{1 - q_{m+1}^n} u'(t_{n+m+2}) \le q_{m+1}^n u'(t_{n+m+1}). \tag{3.37}$$

Combining (3.34) and (3.37) we get that

$$\frac{q_{m+1}^n}{1 - q_{m+1}^n} u'(t_{n+m+2}) \le u(t_{n+m+1}) b_{n+m+1}. \tag{3.38}$$

Inequalities (3.36) and (3.38) show that (3.29) and (3.30) are valid for $k = m + 1$, and therefore (3.29) and (3.30) are valid for all $k \in N$. In particular, (3.29) is valid for all $k \in N$, thus, according to Theorem 1, the equation

$$u'' = -g_1(t)u$$

is nonoscillatory, which contradicts the hypothesis of Corollary 2(i); hence (3.25) is oscillatory.

(ii) The proof is similar to (i), so we omit it.

The proof of Corollary 2 is accomplished. ∎

4. CASE $t_n = t_0 + \lambda^{n-1}T$

In this section we consider the case of $t_n = t_0 + \lambda^{n-1}T$ with $t_0 \geq 0$, $\lambda > 1$, $T > 0$, and denote $\alpha_0 = \lambda + 1 - 2\sqrt{\lambda}$. For this case, we establish the following results.

THEOREM 2. *If there exists $n_0 \in N$ such that, with a_n from* (1.1),

$$a_n \leq \frac{\alpha_0}{\lambda^{n-1}(\lambda - 1)T}, \qquad \text{for } n \geq n_0, \tag{4.1}$$

then (1.2) *is nonoscillatory.*

THEOREM 3. *Suppose that there exist $\alpha > \alpha_0$ and $n_0 \in N$ such that*

$$a_{n_0+k} \geq \frac{\alpha}{\lambda^{n_0+k-1}(\lambda - 1)T}, \qquad \text{for all } k \in N, \tag{4.2}$$

then (1.2) *is oscillatory.*

Proof of Theorem 2. Denote $b_n = \alpha_0/\lambda^{n-1}(\lambda - 1)T$, and define the sequence $\{q_k\}_{k=1}^{\infty}$ by induction

$$q_1 = b_{n_0+1}\left(t_{n_0+1} - t_{n_0}\right) = \frac{\alpha_0}{\lambda}$$

$$q_{k+1} = \frac{b_{n_0+k+1}}{b_{n_0+k}}\left(\frac{q_k}{1 - q_k} + b_{n_0+k}\left(t_{n_0+k+1} - t_{n_0+k}\right)\right)$$

$$= \frac{1}{\lambda}\left(\frac{q_k}{1 - q_k} + \alpha_0\right).$$

Since $\lambda > 1$, hence

$$0 < q_1 = \frac{\alpha_0}{\lambda} = \frac{1}{\lambda}\left(\lambda + 1 - 2\sqrt{\lambda}\right) < \frac{1}{\lambda}\left(\lambda - \sqrt{\lambda}\right) < 1. \tag{4.3}$$

Now notice that the function $f(x) = (1/\lambda)(x/(1-x) + \alpha_0)$ is strictly increasing on $[0, 1)$ and $\alpha_0 = \lambda + 1 - 2\sqrt{\lambda}$. Then

$$\begin{aligned} 0 < \frac{\alpha_0}{\lambda} &= f(0) < f(x) < f\left(\frac{1}{\lambda}\left(\lambda - \sqrt{\lambda}\right)\right) \\ &= \frac{1}{\lambda}\left(\frac{(1/\lambda)\left(\lambda - \sqrt{\lambda}\right)}{1 - (1/\lambda)\left(\lambda - \sqrt{\lambda}\right)} + \lambda + 1 - 2\sqrt{\lambda}\right) \\ &= \frac{1}{\lambda}\left(\lambda - \sqrt{\lambda}\right), \qquad \text{for } 0 < x < \frac{1}{\lambda}\left(\lambda - \sqrt{\lambda}\right). \end{aligned} \tag{4.4}$$

Then (4.3) and (4.4) imply

$$0 < f(0) < f(q_1) < \frac{1}{\lambda}\left(\lambda - \sqrt{\lambda}\right). \tag{4.5}$$

Notice that $q_1 = \alpha_0/\lambda = f(0)$ and $q_2 = (1/\lambda)(q_1/(1-q_1) + \alpha_0) = f(q_1)$. Then (4.5) can be written as

$$0 < q_1 < q_2 < \frac{1}{\lambda}\left(\lambda - \sqrt{\lambda}\right). \tag{4.6}$$

Using the increasing of $f(x) = (1/\lambda)(x/(1-x) + \alpha_0)$, and noticing that $q_{k+1} = f(q_k)$ and relations (4.4) and (4.6), it is easy to show (by induction) that

$$0 < q_1 < q_2 < \cdots < q_k < \cdots < \frac{1}{\lambda}\left(\lambda - \sqrt{\lambda}\right).$$

Now, since $0 < a_{n_0+k} \le b_{n_0+k}$ by (4.1) and the definition of b_n, then from Lemma 2 we get

$$0 < P_k^{n_0} \le \frac{a_{n_0+k}}{\beta_{n_0+k}} q_k \le q_k < \frac{1}{\lambda}\left(\lambda - \sqrt{\lambda}\right), \qquad \text{for all } k \in N,$$

hence (1.2) is nonoscillatory by Theorem 1, and Theorem 2 is proved. ∎

Proof of Theorem 3. If not, then (1.2) is nonoscillatory by Theorem 1, and there is $n \geq n_0$ such that $S_n \leq 1$, i.e.,

$$0 < P_k^n < 1, \qquad \text{for all } k \in N.$$

Denote $b_k = \alpha/\lambda^{k-1}(\lambda - 1)T$, $k = 1, 2, \ldots,$ and define the sequence $\{q_k\}_{k=1}^{\infty}$ to be the same as the one in the proof of Theorem 2 with α, n in place of α_0 and n_0, respectively.

Since $b_{n+k} \leq a_{n+k}$, hence $0 < q_k \leq P_k^n$ for all $k \in N$ by Lemma 2, that is,

$$0 < q_1 = \frac{\alpha}{\lambda} < 1, \tag{4.7}$$

and

$$0 < q_{k+1} = \frac{1}{\lambda}\left(\frac{q_k}{1 - q_k} + \alpha\right) < 1, \qquad \text{for all } k \in N. \tag{4.8}$$

Since $f(x) = (1/\lambda)(x/(1 - x) + \alpha)$ is increasing on $[0, 1)$ and $f(\lambda/(1 + \lambda)) = 1 + \alpha/\lambda > 1$ because of $\alpha > \alpha_0 > 0$ and $\lambda > 1$, hence (4.8) implies that

$$0 < q_k < \frac{\lambda}{1 + \lambda}, \qquad \text{for all } k \in N.$$

In view of the fact that $f(x) = (1/\lambda)(x/(1 - x) + \alpha)$ is increasing on $[0, 1)$ and the fact that $q_1 = f(0) > 0$, it is easy to show that

$$0 < q_1 < q_2 < \cdots < q_k < \cdots < \frac{\lambda}{1 + \lambda},$$

which implies the sequence $\{q_k\}_{k=1}^{\infty}$ has a limit q with $0 < q \leq \lambda/(1 + \lambda) < 1$.

Let k go to infinity in (4.8). We obtain

$$q = \frac{1}{\lambda}\left(\frac{q}{1 - q} + \alpha\right),$$

i.e.,

$$\lambda q^2 + (1 - \lambda - \alpha)q + \alpha = 0, \tag{4.9}$$

which shows that $q \in (0, \lambda/(1 + \lambda))$ is a real root of the equation

$$\lambda x^2 + (1 - \lambda - \alpha)x + \alpha = 0. \tag{4.10}$$

But on the other hand, according to the condition of $\alpha > \alpha_0 = \lambda + 1 - 2\sqrt{\lambda}$ and (4.7), the discriminant of the quadratic form in Eq. (4.10) is

$$\Delta = \left(\alpha - \left(1 + \lambda - 2\sqrt{\lambda}\right)\right)\left(\alpha - \left(\lambda + 1 + 2\sqrt{\lambda}\right)\right) < 0,$$

which shows that Eq. (4.10) has no real root. This contradiction shows that (1.2) is oscillatory, and Theorem 3 is proved. ∎

5. CASE $t_n = t_0 + nT$

In this section, let always $t_0 \geq 0$, $T > 0$, $t_n = t_0 + nT$, $n = 1, 2, \ldots$.

THEOREM 4. *If there is $n_0 \in N$ such that $a_n \leq 1/4n(n+1)T$ for $n \geq n_0$, then Eq. (1.2) is nonoscillatory.*

Proof. Denote $b_n = 1/4n(n+1)T$, and define the sequence $\{q_k\}_{k=1}^{\infty}$ by induction

$$q_1 = b_1(t_1 - t_0) = \frac{1}{4 \cdot 2},$$

$$q_{k+1} = \frac{b_{k+1}}{b_k}\left(\frac{q_k}{1 - q_k} + b_k(t_{k+1} - t_k)\right)$$

$$= \frac{k}{k+2}\left(\frac{q_k}{1 - q_k} + \frac{1}{4k(k+1)}\right).$$

Since $f_k(x) = (k/(k+2))(x/(1-x) + 1/4k(k+1))$ is strictly increasing on $[0, 1)$, and

$$f_k\left(\frac{1}{2(k+1)}\right) = \frac{1}{2(k+2)} \cdot \frac{4k^2 + 6k + 1}{4k^2 + 6k + 2} < \frac{1}{2(k+2)},$$

which can be verified by direct calculation, hence

$$0 < \frac{k}{k+2}\left(\frac{x}{1-x} + \frac{1}{4k(k+1)}\right) < \frac{1}{2(k+2)},$$

$$\text{for } 0 < x < \frac{1}{2(k+1)}, k \in N.$$

Now notice that $0 < q_1 < 1/2 \cdot 2$. Then it is not difficult to show by induction that

$$0 < q_k < \frac{1}{2(k+1)}, \qquad \text{for all } k \in N.$$

According to Lemma 1, for all $n \in N$ we also have $0 < q_k^n < 1$ for all $k \in N$, where the definition of $\{q_k^n\}$ is similar to the one in the proof of Corollary 1(i). Since $0 < a_n \le b_n$ for $n \ge n_0$, it follows from Lemma 2 that $0 < P_k^n \le q_k^n < 1$ for all $k \in N$ when n is large, hence (1.2) is nonoscillatory by Theorem 1. Thus, the proof of Theorem 4 is accomplished. ∎

COROLLARY 3 [2]. *Let $h(t)$ be a nonnegative continuous function on $[0, \infty)$. If $h(t) \le 1/4t^2$ for large t, then*

$$u'' = -h(t)u \tag{5.1}$$

is nonoscillatory.

Proof. Let $b_n = \int_{n+1}^{n+2} h(t)\,dt$. Then $b_n \le 1/4(n+1)(n+2)$, for large n. Let

$$g(t) = \sum_{n=1}^{\infty} b_n \delta(t - (n+2)).$$

Then equation

$$u'' = -g(t)u$$

is nonoscillatory by Theorem 4; therefore Eq. (5.1) is nonoscillatory by Corollary 2(ii), and the proof of Corollary 3 is accomplished. ∎

THEOREM 5. *Let $\alpha > 1/4$. If there is $n_0 \in N$ such that*

$$a_n \ge \frac{\alpha}{n(n+1)T}, \qquad \textit{for } n \ge n_0,$$

then (1.2) *is oscillatory.*

Proof. Choose a number λ with $1/4 < \lambda < \alpha$. According to Theorem 7.1 of [2, Chap. XI], the equation

$$u'' = -\frac{\lambda^2}{t^2}u, \qquad t > 0 \tag{5.2}$$

is oscillatory. Let

$$b_n = \int_{t_n}^{t_{n+1}} \frac{\lambda}{t^2}\,dt = \frac{\lambda T}{(t_0 + nT)(t_0 + (n+1)T)},$$

$$g(t) = \sum_{n=1}^{\infty} b_n \delta(t - t_{n+1}).$$

Then the equation

$$u'' = -g(t)u$$

also is oscillatory by Corollary 2(ii).

Now since $\alpha > \lambda$ implies

$$a_n \geq \frac{\alpha}{n(n+1)T} > \frac{\lambda T}{(t_0 + (n-1)T)(t_0 + nT)} = b_{n-1}, \quad \text{for large } n \in N,$$

hence Eq. (1.2) is oscillatory by Corollary 1(ii).

The proof of Theorem 5 is accomplished. ∎

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